

MODELING OF HUMAN SOCIETY AS A LOCALLY INTERACTING PRODUCT-POTENTIAL NETWORKS OF AUTOMATON

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ABSTRACT. The central problems in social sciences concern the social and psychological mechanisms and conditions required for the emergence and stability of human groups. The present article is dedicated to the problem of stability of human groups. We model human groups using local interacting systems of automaton with relations and reactions and using the structural balance theory. The ‘structural balance theory’ ties the emergence of a human group with the human actor’s thoughts about how another actor treats him and his perception of actors. The Cartwright and Harary formalization the concept of balance theory within a graph theoretical setting unable to get a number of mathematical results pertaining to an algebraic formulation of the theory of balance in signed networks/graphs. The deeper generalization of ‘balance theory’ as the smooth product-potential fields on domain gives us the ability to create theory of ‘smooth product potential social fields’. We then find that all discrete product-potential system tightly connect with other process - process multiplication on the randomly chosen matrices and we find connections between stationary measures and some algebraic objects.

Keywords: social systems, balanced groups, potential fields, networks of automata, product integral, ideals, Markov chains.

Introduction

The present article is dedicated to the problem of stability of human groups, firstly by modeling human group using locally interacting systems of automaton with relations and reactions. We will concentrate our attention on the particular type of local interacting systems of automata named the product potential social system. The product potential social system is a mathematical analogy of balance in the social sense human group. The problem of stability has a social source. The source of mathematical realization is the cellular automata theory and the theory of interacting systems.

Homans and group dynamics(1958,1974). Homans defines the social structure of the group as an equilibrium state of social system (he defines the term “group”, too). This is a good way of linking dynamics and structure. He wants to specify the mechanism that produce and maintain a social structure. He speculates that the process would be described by the system of differential equations that would have equilibrium solutions and these solution would be those states of social system that are the social structure.

He distinguishes an “internal system” and “external system”. It means that any social system has two functional problem: adapting to an environment and integrating its units. He asserted that the more frequently persons interact, the more similar they become in sentiments. But he had not really modelled the process of social interaction that would yield this property. To rectified this deficiency Homans created the conception of “social behavior as exchange”. Social interaction is an exchange involving such the key variables as value of activities and frequency of activity. The Homan’s exchange theory and an exchange in the competitive market economics are based on the very same behavioral exchange.

Coleman had studied the exchange formulated by Homans and he (1990) produced his own version of this idea. This work uses a general equilibrium framework from competitive market economics as it did Homans. Coleman treats problem such as emergence of norms, interpersonal comparison of utilities, and the derivation of the interests of collectivity from the exchange among its part. Coleman's model start from two relations connecting two sets of entities: actors and resources (events). Each actor has some level of interest in each resource and also some proportion of control over each resource. Actors are willing to give up some control over some resources to gain more control over other resources of interest. The exchange occur via maximization of utility by each actor subject to a constraint expressed in terms the given. The interests are expressed in terms of structure in parametric role and the state of the system is described in terms of distribution. So, they take a structural form as given.

The Heider's (1946,1958) balance theory. Heider considered very simple relationship among two or three person. According to Heider, balance is not a real correction of forces among elements but only perception of them by a person. If one actor thinks that another actor treats him well, any negative act "falls out" from hole picture. The cause is the mental forces that aspire to restore equilibrium. From the "behavioral" point of view the most important consequence of the Heider and his followers is a supposition that: (1) positive attitude is transitive ("I like that person, whom my friend like") ; (2) negative attitude is non-transitive (the following principle does not work: "I hate the person whom my enemy hates").

Newcomb (1953) extends Heider's ideas to the system interaction in which the standpoints of all interactants are jointly considered.

Cartwright and Harary (1956) formalized the concept of balance theory within a graph theoretical setting. A circuit consisting of two or more edges is a positive if circuit has an even number of negative relations (in this model all relation may be only positive or negative). A graph of relations is considered to be balanced if all circuits consisting of two or more edges are positive. Cartwright and Harary proved the important **Structure Theorem**:

The graph of relation is balanced if and only if it can be partitioned into two subsets (one of which may be empty), such that all positive cycles are within the two subsets and all of negative paths are between them.

Davis (1967) generalized the structure theorem, shoving that balance holds, the structure breaks up into a finite number of subgroups with positive relationships within subgroups and negative relationships between them. Davis suggested the next model of clustering: a graph is clustered if no circuit includes exactly one negative edge. In the model of ordered cluster Davis and Leinhardt (1972) is supposed the existence of a hierarchy of subgroups in which every level includes at least one subgroup (in this model all relation may be only positive or neutral).

Next very important problem is what kind **mathematical tools** were used for mathematical modeling of social processes. Cartwright and Harary (1956) and Davis (1967) formalized the concept of balance theory by using a graph theory. Simon (1957) have used the differential equations method. Harrison White (1963) worked out the abstract algebraic group method for classification kinship system. Hunter (1978) used the differential equations method. Fararo and Scvoretz, (so called E-state model, 1986) have used the stochastic concept. Kovchegov (1987, 1994) worked out the thermodynamics formalism for classification of

stable social system. Recently, the 'structural balance theory' research has received renewed attention starting with Doreian and Mrvar (1996).

We can say that only Hunter (1978) tried to fulfill the Homans plan. He use four social-psychological mechanisms: influence (my feeling toward you will be more positive to the extent that my friends say nice thing about you or to the extent that my enemies that my enemies say bad thing about you), compatibility (my feeling toward you will be positively affected if you say nice thing about my friends and nasty thing about my enemies), carryover (if I like you, my feelings will tend to become more positive since I will give you "the benefit of the doubt" on all ambiguous statement. If I dislike you, everything will be just the opposite), and reciprocity (my feelings will become more positive if you say nice thing about me and I will react negatively when you say negative things about me). But, as is easy shown, the measure of the non-balanced invariant under Hunters' procedure sociomatrices is greater or equal then measure of balanced sociomatrices.

John von Neumann invented the **cellular automata** in the late 40's and early 50's (1945, 1949, 1951) when he sought to investigate the question of life's origin by trying to design a self-reproducing automaton. The theory of cellular automata (Burks, 1966, Codd 1968) is an example of a deterministic system with local interactions. Every automaton resides in an integer point of two or more dimensional real space and has a state (normally 0 or 1). The system has an initial distribution of states. The dynamics of the cellular system is defined by the system of rules. The state of an automaton in a given moment of time t depends on the state of the neighbors in the previous moment $t-1$ (time is discrete). The rules are defined for all possible combinations of states of neighbors. The cellular automata are a homogeneous system: all automata use the same rules. Every rule is a prescription that assigns a particular value of state for the given automaton for every particular distribution of states of neighbors.

The theory of interacting systems emerged as a branch from the theory of probability towards the end of the 60's. Historically, the first problems that motivated people to pay attention to interacting particle systems were problems of statistical physics particularly the Ising model (Glauber, 1969, Dobrushin, 1971). The problem was to construct thermodynamics descriptions of the evolution of systems where the states have the classic Gibbs measures and then find the phase transitions.

Very soon, other sciences found similar problems to those found in the theory of local interacting systems. We will only list the most popular models, where the theory of local interacting system is used. Clifford and Sudbury, 1973 and Holley and Liggett, 1975 then did elaborations to the voter model (behavioral science). The local interaction was defined on the integer points of n dimensional space. Every voter (Holley and Liggett interpretation for a voter model) can support one of two political parties, denoted by 0 or 1. A voter located on the integer points of n dimensional space with non-zero probability get the position of spatial neighbors. Clifford and Sudbury gave the following biological interpretation: two populations denoted by 0 or 1, fight for territory. The voter model state of the system then contains the field 0 or 1.

The **thermodynamics description** is the description of systems that uses the unit of **measure as the state of system**. Thus, the thermodynamic state for the system is measure.

For the voter model, there exist at least two invariant measures: the delta measure on the system states in which all people are located in the same position, and secondly, for 3

or more dimensional space the existence of additional families of invariant measures. The contamination model (Harris, 1974) is a model of the spread of infections, where healthy people surrounded by contaminated people get infections with non-zero probability. A sick person recovers with a non-zero probability as well.

The profound presentation of these models and other models are contained in book Liggett (1985).

Kovchegov 1984, 1994 elaborated the model of a human society using nets of automata with relation and reactions. This model belongs to the class of local interacting systems. The dynamics on the nets of automata with relation and reactions was defined by interacting with neighbors, where neighbors are neighbors on a graph of relations (not spatial). The difference between the previous model and the current model of a human society is the existence of relations and psychological reactions that adjust the perception of a neighbors' state. This adjustment depends on the state of relations. This difference is crucial in that essentially the system is non-ergodic. The system is called ergodic if all initial measures of the system converge to a unique invariant measure. The social system will be balanced (in the social meaning) if the system has maximum invariant measures (maximum taken for all possible relations or fields of psychological reactions on the edges of the graph of relations). The balanced social system satisfies the principle of maximal non-ergodicity. All fields of psychological reactions that provide a balanced (in the social sense) social system are product potential fields. A product-potential system is a mathematical analogue of the social balanced system.

In the temporary theory of automata and computer science, the term "automaton" is used in formal systems that have internal sets and transform input words into output words. This automaton has input and output alphabets, the set of internal states, and the set of rules (programs). The rules create current symbols for output words as a function of the current input word and changes the internal state. For deterministic automaton the choosing of a particular rule depends only on the internal current state. For probabilistic automaton this choice depends on the state and is performed randomly.

We, however, do not use this term in the sense of theory of automata. Our meaning of the term "automaton" is more similar to the meaning of "cellular automaton" in the cellular automata theory. There the "automaton" can perform a few actions as well. In this article the automaton can only perform two actions: random choice and psychological adjustment. Our automaton can randomly choose a neighbor, but cannot transform an input word into an output word.

So when we use the term "probability automata" this means that we have only used the property to make a random choice. But we enable our automaton with some number of additional actions for future modeling of collective actions. Part four contains an example of this type of application. The ability to make a large number of different actions (not limited to psychological adjustments) is the main reason why we use the word "automaton" instead of the word "element" the way it is used in the theory of local interacting systems.

The automaton models that were done in part on the structural balance theory from Heider, Cartwright and Harary, Davis and the cognitive theory of Festinger. Festinger's cognitive dissonance theory (Festinger, 1957) allows us to formalize the process of motivation. Within the framework of this model an attempt to formalize the notion of dissonance - the main notation of Festinger's theory - was undertaken.

It is supposed that every automaton at every moment is in a certain state that belongs to the finite set of state. In addition, every automaton is endowed with a set of "psychological" reaction that form an algebraic group of permutations of the finite of the automaton state. According the dissonance theory each automaton has the ability to make "psychological" adjustment it's perception of a neighbor in the graph relation. The adjustment depend on the feeling of the choosing automaton toward the chosen automata. The following principle was used: if my "enemy" is feeling well then I'm feeling bad, and if he/she feeling bad then I'm feeling well. For the "friend" the relation is opposite.

For this class of models a structural form is given. It means that a structure (the graph of relation) was put in parametric role and the state of the system was described in terms of distribution of the states of actors. Then for every graph of relation (social structure) was constructed the finite discrete Markov chain. Every Markov chain has set of invariant measures. The following criterion for selection of the balanced structures was chosen (the principal of maximum nonergodicity)

"We must select the structures where the associated Markov chains have the maximum number of stationary measures (the maximum taken over all structure of relation)".

It means that only group of actors with the structures of relation satisfied to the principal of maximum nonergodicity will be survival. We can find the groups of reaction when the selected structures will be balanced according Cartwright and Harary or Davis.

This article has three parts. Part one contains description of theory of locally interrupted system with relations and reactions. The potential networks were deduced from locally interrupting system by using so call "principle of maximum nonergodicity". The system are balanced in social sense if the set of psychological reactions on the graph of relation satisfy the principle of maximum nonergodicity and this system of reactions are product potential on the so call "two-steps" graph of relations. In real life survive only relatively stable groups and we can observe only stable groups that in social science call balanced. So reason why social system (group) is stable based on hidden potentiality of reactions: only social systems with potential system of reactions (potential fields) are stable (balanced in social sense).

The main problem what will prove in part two is problem of existence of potential fields. For this purpose we use method of smooth fields on the solid domain and product integrals. For of smooth fields will be write the system of infinitesimal equations (system of partial differential equations) that must be hold for all potential fields. The system of partial differential equations can be transformed into the system linear differential equation with one additional condition: the matrix-solution and field have to be anti-commutative pair. Then was found solutions of infinitesimal equations: the solution is any parameterizations of intersections of intersection of second-degree surface (set of matrices that $A^2=E$) and arbitrary plane. The set of solutions can be represented in few canonical forms. Then property to be potential will be checked by computer calculations.

Then potential field on domain will be transformed into discrete potential marking on embedded graph of relation by using product integrals. The finally will be found system differential equations that transfer any initial fields of reactions into potential.

A locally interacting process for the product potential system of relations can be given by an algebraic representation of an process of multiplication on the randomly chosen so call "control matrix". In part three we found one to one maps between thermodynamic states of system (the thermodynamic state for the system is measure) and so call "left ideals" on the semi-group of control matrices. The ideal matrices have a very important property: when

an arbitrary stochastic/control matrix is multiplied from the left by an ideal matrix one obtains a left ideal matrix. So the set of left ideal matrices is the termination set for our stochastic product process (spatial Markov's chain). It means that once the system reaches the termination set the process can never leave the termination set. Thus the left ideal matrices play a crucial role in the description of our process.

Part 1. THE GENERAL THEORY OF THE LOCALLY INTERACTING NETS OF THE AUTOMATA.

In the first part of paper we will describe the general model of locally interacting network of automaton.

1.1 Formulation of the automaton model of human groups .

The next object is called the **model of human groups with relations**

$U = \{ \Gamma = (A, B); Y \text{ and } f: B \rightarrow Y; E_i \text{ and } G_i, \forall i \in A; \psi_i: Y \rightarrow 2^{G_i}, \forall i \in A; Z, \Pi, \text{Act}: \Pi \rightarrow Z \}$

The graph of relations

Automaton model of the human society is a network of automata connected to each other by a graph of relations $\Gamma = (A, B)$, where A is a set of the vertices, B is a set of the edges encoding the existence of relations among the members of the group. It is supposed that Γ is connected, directed finite graph without loops and if $(i, j) \in B$, then $(j, i) \in B$.

Example. A complete directed graph $\Gamma_3: A = \{1, 2, 3\}$, $B = \{(i, j), \forall i, j \in \{1, 2, 3\}, i \neq j\}$.

A set of states of relations and a function of relations . Let Y be a set of states of relations among members of the group and $f = \{f(i, j), \text{for } \forall (i, j) \in B, f(i, j) \in Y\}$ is the structure of relations. The relations are symmetrical if $f(i, j) = f(j, i) \forall (i, j) \in B$.

Example. $Y = \{+1, -1\}$ is the set of states, $y_{i,j} = f(i, j) = +1$ if automaton i has positive "relation" to automaton j and $y_{i,j} = f(i, j) = -1$ if the "relation" is negative.

The set of states of automaton and a group of reactions. It is supposed that every automaton at every moment is in a certain state that belongs to the set of states E . In addition, every automaton is endowed with a set of "psychological" reactions that form an algebraic group G and are realized as transformation E (permutation for finite E) on itself, i.e. $\forall g \in G \text{ is } g: E \rightarrow E$.

Example. $X_i = X = \{+1, -1\}$, where the state signed by symbol "+1" may be interpreted as positive and "-1" - as negative.

Let $G_i = G = \{g, e\}$ is group and it is realized as substitutions of the set of states $X = \{+1, -1\}$:

$$g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } e = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$$

i.e. $g(x) = -x, e(x) = x, \forall x \in X$ and $g^2 = e$.

The last property ($g^2 = e$) is very important for social modeling . This property can be interpreted as a model of the law of logic called the **double negation** . It's the main reason why we must demand that this property holds for all psychological reactions. It means that for any algebraic group G this property must hold, i.e. for all $g \in G$ $g^2 = e$ holds.

The choice function.

Every automaton has the ability to make a "psychological" adjustment of it's perception of a neighbor in the graph of relations. The adjustment depends on the feeling of the chosen automaton towards the other one. And it is given by the "choice" function $\psi_i(f(i,j)) \in G$, for $\forall i \in A$ and $\forall (i,j) \in B$.

Example. $\forall i \psi_i(-1) = g, \psi_i(+1) = e$

Consequently if automaton i has bad feelings toward automaton j , i.e. $f(i,j) < 0$, which means he is "enemy" of j , then the state of the second one is received by the first one after adjustment on reactions as $x_i = \psi_i(f(i,j))x_j = \psi_i(-1)x_j = gx_j = -x_j$. For the "friend" ($f(i,j) > 0$) the relation is opposite.

The set of actions and Act - function.

Let Z be a set of actions; the set Z has very complicate structure: it contains, for example, the actions (operators) which can be done by one human being, by two, by three and so on. Every operator has it's own domain. Let $\Pi = \{E_1, E_2, \dots, E_n\}$ be a partition of the set of states E (it means that next two equalities $E = E_1 \cup E_2 \cup \dots \cup E_n$ and $E_i \cap E_j = 0$, hold for all $i \neq j$). The function $Act : \Pi \rightarrow Z$ is called the action function. If, for example, $x \in E_i$, then $Act(x)$ is the set of possible actions (operators) from the set of actions Z , which required i people for realization. In case when our group is a discrete discontinuous group and has the fundamental domain we can transform an arbitrary partition of the fundamental domain to a partition of the whole E . The *normal partition* is the partition which is generated by the partition of the fundamental domain.

Example . Let $G = \{f, g\}$ be a group which is generated by two elements $f(z) = 2z$ and $g(z) = \frac{3z+4}{2z+3}$. The group G transforms the set $E = H^2 = \{z = x + iy, \text{ where } x \text{ is positive real number}\}$ onto H^2 . We easily see that this group has a convex fundamental (the hyperbolic quadrilateral) domain D and an arbitrary partition of D generates the normal partition of E by the group of transformations G .

Note that the Möbius group $G = \{f, g\}$ is generated by four reflections with respect to any side of the fundamental polygon which contains exactly four sides (all sides are circles). But for a reflection the double negation property holds, it means that if we map our space H^2 two times by reflection, then we receive the identity function (the identity function on H^2 is the function that maps each element of H^2 to itself).

Suppose that the set of state of relationship Y is a finite linearly ordered set and $Y = \{y_1, y_2, \dots, y_n\}$, where $y_i R y_j$ if and only if $i < j$. Let $A(y_i) = y_{i+1}$, if $i \leq n-1$, $A(y_n) = y_n$ and $B(y_i) = y_{i-1}$, if $i > 1$, $B(y_1) = y_1$. If $E = E_{-m} \cup E_{-m+1} \cup \dots \cup E_{-1} \cup E_0 \cup E_1 \cup \dots \cup E_{m-1} \cup E_m$, then for $x \in E_k$ $Act(x) = A^k$ and for $x \in E_{-k}$ $Act(x) = B^k$, where $0 \leq k \leq m$. It means that a person can make relation with another person better or worse by operators A and B . In this case our reactions $g_{i,j} \in \Psi(y_{i,j})$, where $y_{i,j} = Act(x_i(t-1))$ belong to the set of actions Z too.

The *state of a system* is the function $x = \{x_i, \text{ for all } i \in A \text{ and } x_i \in E\}$. Let W be the set of states of the system.

The marking of the edges of the graph Γ by reactions is given by the set $R(G) = \{g_{ij}, \forall (i,j) \in B\}$. We consider, as a rule, those marking $R(G)$ for which there exist a function of relation f such that $g_{ij} \in \psi_i(f(i,j)) \forall i \in A$ for any $j \in \partial\{i\}$, where $\partial\{i\}$ is the set of neighbors of the automaton i on Γ . When it is easy to understand what kind of group is used we will use the notation R , i.e. the symbols R and $R(G)$ have the same meaning. At the same

time we will use a "two-steps graph" $\Gamma^* : \Gamma^* = (A^*, B^*)$, where $A^* = A$ and $(i, j) \in B^*$ if and only if there is $k \in B$ such that $(i, k), (k, j) \in B$.

The marking of the "two-steps graph" Γ^* by reactions $a_{i,l}(k) = g_{k,i}^{-1}g_{k,j}$ will be denote by symbol R^* . It means that $R^* = \{ a_{i,j}(k) , \forall (i,j) \in B^* \text{ and for all admissible } k \}$, it means for all k that there is a path $\{ (i,k), (k,j) \} \subset \Gamma$.

Now we can define a "product integral" or P-integral along the way on the Γ (Γ^*).

If $L_{i,j} = \{ (i_1, i_2), (i_2, j_3), \dots, (i_{n-1}, i_n) \}$ is an arbitrary directed way from i to j on the graph Γ (where $i_1 = i, i_n = j$), then a product integral along the directed way $L_{i,j}$ is

$$R(L_{i,j}) = g_{i_1, i_2} g_{i_2, i_3} \dots g_{i_{n-1}, i_n}.$$

If $L_{i,j}^*$ is an arbitrary directed way on the graph Γ^* from i to the j , $L_{i,j}^* = \{ (i_1, i_2, i_3), (i_3, i_4, i_5), \dots, (i_{2k-1}, i_{2k}, i_{2k+1}) \}$, where (i, k, j) is the edge (i, j) of the graph Γ^* that corresponds to the path $\{ (i, k), (k, j) \}$ on the graph Γ and $i_1 = i, i_{2k+1} = j$, then a product integral under the directed way on the "two-steps graph" is

$$R(L_{i,j}^*) = a_{i_1, i_3}(i_2) \dots a_{i_{2k-1}, i_{2k+1}}(i_{2k}).$$

Example 1. Suppose we are given a complete, therefore not bipartite, graph of relations Γ_3 and its two-steps graph Γ^* (see figure 1). The way $L = \{ (1, 2), (2, 3), (3, 1) \}$ is the close directed way on the graph Γ_3 and way $L^* = \{ (1, 2, 3), (3, 1, 2), (2, 3, 1) \}$ is the close way on the graph Γ^* . The way L^* on the graph Γ^* is induced by the way L on the graph Γ . We easily can figure out the product integrals $R(L)$ and $R(L^*)$:

$$R(L) = R(\{(1, 2), (2, 3), (3, 1)\}) = g_{1,2}g_{2,3}g_{3,1}$$

$$R(L^*) = R(\{(1, 2, 3), (3, 1, 2), (2, 3, 1)\}) = a_{1,3}(2)a_{3,2}(1)a_{2,1}(3)$$

2. Suppose we have the bipartite graph Γ and its two-steps graph Γ^* . We easily see that a graph Γ^* has two connecting components (it's true for all bipartite graph). Let $L = \{ (1, 2), (2, 3), (3, 4), (4, 1) \}$ be the close directed path on the Γ , and let $L_1^* = \{ (1, 2, 3), (3, 2, 1) \}$ and $L_2^* = \{ (2, 3, 4), (4, 1, 2) \}$ be two different paths, on the two different components of the graph Γ^* , that were induced by way L .

Similarly we find the all multiplicative integrals:

$$R(L) = g_{1,2}g_{2,3}g_{3,4}g_{4,1} \quad R(L_1^*) = a_{1,3}(2)a_{3,1}(2) \quad R(L_2^*) = a_{2,4}(3)a_{4,2}(1)$$

The dynamics of the states of the system is determined by family of conditional probabilities $Q = \{ q_i(x|x_j, \forall j \in \partial\{i\}), \forall i \in A \}$. It is supposed that

$$q_i(x|x_j, \forall j \in \partial\{i\}) > 0$$

if and only if $x \in [x_j, \forall j \in \partial\{i\}]$, where $[x_1, \dots, x_m]$ is non-ordered set, sublattice and so on if E is a lattice.

If the family of conditional probability Q is fixed, then on the set of probability measures M a Markov operator

$$\mu Q(x) = \sum_{y \in F^{-1}(x)} \prod_{i \in A} q_i(y_i | g_{ij}x_j, \forall j \in \partial\{i\}) \quad (1)$$

is defined for any structure of relations f , where μ is an arbitrary probability measure, $\mu \in M$. Here $F^{-1} = \{ y \in W : x \in F(y) \}$,

$$F(x) = \prod_{i \in A} [g_{ij}(x_j) \forall j \in \partial\{i\}]$$

1.2 The product potential (non-dissonant) system.

Let $Y = \{y_{i,j}, \forall (i,j) \in B\}$ be the structure of relations. The system of reactions $R = \{g_{i,j}, \forall (i,j) \in B \text{ and } g_{i,j} \in \psi_i(y_{i,j})\}$ is called *potential (non-dissonant)* if there exists a state of the system $x \in W$ that $\forall i \in A$

$$g_{i,j}x_i = g_{i,k}x_k, \quad \forall j, k \in \partial\{i\}$$

hold.

Let W_0^R be the set of non-dissonant states of the system.

If $F(W_0^R) \subseteq W_0^R$ then the non-dissonant structure of relations remains non-dissonant for some time.

When W_0^R is not empty and $F(W_0^R) \subseteq W_0^R$?

The potential system in physics .

The vector field $\overline{F}(x, y) = (P(x, y), Q(x, y))$ is called a potential if and only if

$$\int_{L_A} P(x, y)dx + Q(x, y)dy = 0 \quad (*)$$

holds for all closed paths from a point A to point A.

If condition (*) holds, we can define a function (a potential function)

$$u(B) = \int_{L_{A,B}^1} Pdx + Qdy = \int_{L_{A,B}^2} Pdx + Qdy$$

for any two paths from a fixed point A to an arbitrary point B.

In this case the equalities

$$\overline{F}(x, y) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$

and $du = P(x,y)dx + Q(x,y)dy$ hold.

In our case we have similar situation. The "product integral "

$$R(L_{i,i}) = e \quad (R(L_{i,i}^*) = e)$$

for any closed path $L_{i,i}$ ($L_{i,i}^*$) on the graph Γ (Γ^*) (we can compare this property ,which was called property A1 before with property (*)). The potential function is

$$u(k) = r_{i,k} = R(L_{i,k}^1) = R(L_{i,k}^2) \quad (u(k) = r_{i,k} = R(L_{i,k}^{1*}) = R(L_{i,k}^{2*}))$$

, for the arbitrary paths $L_{i,k}^1$ ($L_{i,k}^{1*}$) and $L_{i,k}^2$ ($L_{i,k}^{2*}$) from a fixed vertex i to an arbitrary vertex k on the graph Γ (Γ^*) .

1.3 The main system of the potential equations.

The system of equations

$$R(L_{i,i}) = e \quad \text{for all } i \in A \quad (R(L_{i,i}^*) = e \quad \text{for all } i \in A)$$

for any closed path $L_{i,i}$ ($L_{i,i}^*$) in the graph Γ (Γ^*) is called *the main system of the potential equations* .

A solution of the main system is a list $\{g_{i,j}, \forall (i,j) \in B\}$ that makes each equation a true statement when the elements of the group G $\{g_{i,j}, \forall (i,j) \in B\}$ are substituted in the main system of equations.

Example of solutions for the main system of equations for Γ_3

Let $a_{1,2} = g_{3,1}^{-1}g_{3,1}$, $a_{2,3} = g_{1,2}^{-1}g_{1,3}$, $a_{3,1} = g_{2,3}^{-1}g_{2,1}$.

The main equation is $a_{1,2}a_{2,3}a_{3,1} = e$.

This equation has the family of solutions:

$$g_{1,2} = x_1, g_{3,1} = x_2, g_{3,2} = x_3, g_{2,1} = x_3^{-1}x_2x_1x_3^{-1}x_2, \\ g_{1,3} = x_1x_3^{-1}x_2x_1x_3^{-1}, g_{2,3} = x_3^{-1}x_2x_1x_3^{-1}x_2x_1x_3^{-1},$$

where x_1, x_2 and x_3 are arbitrary elements of group G ($a_{1,2} = x_2^{-1}x_3$, $a_{2,3} = x_3^{-1}x_2x_1x_3^{-1}$, $a_{3,1} = x_3x_1^{-1}$).

This family of solutions is very interesting. They show us that different automata play different roles in the society. The automaton 3 can react arbitrary to the automata 1 and 2 ($g_{3,1} = x_2$, $g_{3,2} = x_3$, where x_2 and x_3 are arbitrary independent variables). The automaton 1 can feel himself free with respect to the automaton 2 ($g_{1,2} = x_1$), but must adapt its behavior with regard to automaton 3. The automaton 2 plays a central role in the process of maintaining a potential (non-dissonant) property in the group. The automaton 2 simulates a very adaptable flexible personality: it has to adapt its behavior with respect to the other two automata.

This interpretation means that a potential system of automata can't be nonhomogeneous.

So the within network of relations arises the problem of existence of potential networks. This problem can get a complete solution for potential fields (markings) with values in set of the 2 by 2 matrices.

The thermodynamics description of nets of automata with relations and group of reactions: the principle of maximum nonergodicity for a finite graph and a finite order group of reactions.

It is natural to suppose that in reality those and only those group structures are observed where the matrix of transition probabilities of the corresponding Markov chain has a limit at $t \rightarrow +\infty$.

Model 1.

In model one ([33]) we have used an existence of limit for transition matrix of Markov's chain as criterion for selection of balanced graph.

Let $P = (p(x,y))$ for any two states of system x and y . Our criterion was demanding of existence of limit $\lim_{n \rightarrow \infty} P^n$. The analysis of model [33] show as that there is connection between existence of limit and stationary points of map F and number of stationary measures ($pP = p$).

The limit exist if and only if W_0^R contains only stationary points of map F ($W_0^R = \{x \mid F(x) = x\}$).

The limit exist if and only if the marking of edges Γ is product-potential field ($R(L_{i,i}) = e$ for any close direct path $L_{i,i}$).

The number of stationary measures of our Markov's chain reach maximum (for all possible relations) only for system of automaton for which exist limit of iterations of transition matrix.

Example. (1) Suppose we have a complete graph of relations Γ_3 .

$Y = \{+1, -1\}$ is the of state of relations. Let $f = \{y_{1,2} = y_{2,3} = +1, y_{3,1} = -1, y_{i,j} = y_{j,i}\}$ be a structure of relations, The group of reaction is $G = \{g, e\}$, where $g(x) = -x$ and $e(x) = x$, for all $x \in E = \{+1, -1\}$. The choice function is $\psi(-1) = g$, $\psi(+1) = e$. Let $R = \{g_{1,2} = g_{2,3} = e, g_{3,1} = g, g_{i,j} = g_{j,i}\}$ be the marking is accorded with the structure of relations f by the choice function ψ , i.e. for all $(i,j) \in B$ $g_{i,j} \in \psi_i(f(i,j))$.

$\forall x = (x_1, x_2, x_3) \in W$ has

$$x \rightarrow F(x) = \{ex_2, gx_3\} \times \{ex_1, ex_3\} \times \{ex_2, gx_1\} = \{x_2, -x_3\} \times \{x_1, x_3\} \times \{x_2, -x_1\}$$

It is mean that for arbitrary state of system $(x_1, x_2, x_3) \rightarrow (x_2, x_1, x_2)$ with probability $q_{1,2} q_{2,1} q_{3,2}$. Similarly $(x_1, x_2, x_3) \rightarrow (x_2, x_1, -x_1)$ with probability $q_{1,2} q_{2,1} q_{3,1}$ and so on. So we have Markov chain with 8 by 8 transition matrix $P = \{ p_{x,y} \text{ for any states of system } x \text{ and } y \}$. The set of states of system contains 8 elements.

Right now we will describe the very important set W of states that F maps into one element set ($| F(x) | = 1$): $W = \{ (x, -x, x), (-x, x, -x) \}$. Really with probability 1 $F: (x, -x, x) \rightarrow (-x, x, -x)$ and $F: (-x, x, -x) \rightarrow (x, -x, x)$. It means that transition matrix is oscillated and limit does not exist. For this system we have only one stationary measure.

(2) Let $R = \{ g_{1,2} = g_{1,3} = g, g_{2,3} = e, g_{i,j} = g_{j,i} \}$.

So system is balanced and field is product potential.

In this case $F: x \rightarrow F(x) = \{-x_2, -x_3\} \times \{-x_1, x_3\} \times \{-x_1, x_2\}$. The solutions of equation $-x_2 = -x_3, -x_1 = x_3, -x_1 = x_2$ is set W . In this case W contains only one family of solution $(x, -x, -x)$ for any x from X . All this solutions are stable points: $F(x, -x, -x) = (x, -x, -x)$. So with probability 1 $F: (x, -x, -x) \rightarrow (x, -x, -x)$.

How easy to check in this case Limit exists and system has two stationary measures ($pP = p$).

(3) Let $R = \{ g_{1,2} = g_{1,3} = g_{2,3} = g, g_{i,j} = g_{j,i} \}$. So system is not balanced and field is not product potential.

In this case $F: x \rightarrow F(x) = \{-x_2, -x_3\} \times \{-x_1, -x_3\} \times \{-x_1, -x_2\}$. The set W is set of solution of system of equations $x_2 = -x_3, -x_1 = -x_3, -x_1 = -x_2$ contains one family of solutions (x, x, x) for all x from X . $F(x, x, x) = (-x, -x, -x)$ for all x from X . So $F(-1, -1, -1) = (-1, -1, -1)$ and $F(-1, -1, -1) = (+1, +1, +1)$ with probability one. It means that transition matrix is oscillated and Limit does not exist.

In this case we have exactly one stationary measure.

Note. The behavior of map F on the set W determines the property of entire system.

Model 2.

In model [35] for selection of balanced groups was used "principle of maximum nonergodicity". For this purpose we use "two-steps" graph and demand potentiality only for field of reactions on the "two-steps" graph Γ^* (see conditions A1 below). It is mean that that initial system of reactions we do not demand to be potential. Instead of demanding potentiality of initial field R we demand for field R that condition A2 (see below) hold.

The marking of the edges of the graph Γ by reactions is given by the set $R = \{g_{ij}, \text{ for all } (i, j) \in B \text{ and where } g_{i,j} \in \psi_i(f(i, j))\}$. We say that the marking R satisfies condition

A1 if for any vertex i and for any closed path $L_{i,i}^*$ in the graph Γ^* the equation $R(L_{i,i}^*) = e$ holds, where e is the unit of the group G .

We say that the marking R satisfies condition

A2 if for any vertex i such that $\partial\{i\}$ includes at least two elements the equalities $g_{ij}g_{ji} = g_{ik}g_{kj} \forall j, k \in \partial\{i\}$ hold.

In this case we call reaction $a_i = g_{i,j}g_{j,i}$ the *characteristic reaction* of the element i . If $a_i = a$ for all $i \in A$ then we call the reaction a *characteristic reaction* of the group.

This trick give us ability find reasonable theorems for calculation of number stationary measure by founding solution of some equations.

The analysis of Model 1 has shown that the criterion for the selection of the structures of relations according to the existence of the limit of the transition probabilities matrix of the associated Markov chain is the same as the criterion for selection of the structures by the maximum number of stationary measures fore associated Markov chains (the maximum

taken over all structures of relations). This last principle of selection, called **the principle of maximum nonergodicity**, will be used here.

Proposition . If for marking R the conditions A1 and A2 hold, then W_0^R isn't empty and $F(W_0^R) \subseteq W_0^R$ and $W_0^R = W_0 = \{x \in W : |F(x)| = 1\}$, where $|F(x)|$ is the number of elements in the set $F(x)$.

If Γ is a graph of relations that is not bipartite then $W_0 = \{z(t), \forall t \in E\}$, where $z(t) = \{x_j = R(L_{j,i}^*)t, \forall j \in A \text{ where } L_{j,i}^* \text{ is a path in } \Gamma\}$, i is an arbitrary fixed vertex and $x_i = t$. If Γ is a bipartite graph then $A = A_1 \cup A_2, A_1 \cap A_2 = \emptyset, \Gamma_i^* = (A_i, B_i^*)$ are two connected components of the unconnected graph $\Gamma^*, i = 1, 2$, and $W_0 = \{z(t, r), \forall t, r \in E\}$, where $z(t, r) = \{x_s = R(L_{s,i}^{(1)*})t, \forall s \in A_1; x_k = R(L_{k,j}^{(2)*})r, \forall j \in A_2\}$; i, j are arbitrary fixed vertices that accordingly belong to A_1, A_2 ; $L_{s,i}^{(1)*}, L_{k,j}^{(2)*}$ are the directed sequences on Γ_1^* and Γ_2^* respectively; $x_i = t, x_j = r$.

Suppose, that there is a positive integer number m such that equality $g^m = e$ holds for all $g \in G$. In this case the set

$$Orbit(\{g\}, x) = \{x, gx, \dots, g^{m-1}x\}$$

is called a $\{g\}$ -orbit of element $x \in E$, where $\{g\}$ is a subgroup of the G , which is generated by g . Let $\bar{x}^0 = (x_1, x_2, \dots, x_n)$ be initial state of the net of automata containing n automata. Let denote by the symbol $E(\bar{x}^0)$ the set that consists of all different elements x_1, x_2, \dots, x_n , i.e. $E(\bar{x}^0) = \{x_1, x_2, \dots, x_n\}$.

Example. Suppose, $\bar{x}^0 = (1, 2, 1, 1, 1)$, then $E(\bar{x}^0) = \{1, 2, 1, 1, 1\} = \{1, 2\}$.

Let $gE(\bar{x}^0) = \{g^k x \text{ for all } x \in E(\bar{x}^0) \text{ and for all integer numbers } k\}$. There is a partition of $gE(\bar{x}^0)$: $\tau_g = \{Orbit(\{g\}, x_i), Orbit(\{g\}, x_l), \dots, Orbit(\{g\}, x_s)\}$, where $gE(\bar{x}^0) = Orbit(\{g\}, x_i) \cup Orbit(\{g\}, x_l) \cup \dots \cup Orbit(\{g\}, x_s)$ and for $i \neq j$ equality $Orbit(\{g\}, x_i) \cap Orbit(\{g\}, x_j) = \emptyset$ holds.

The number of elements (orbits) in the partition τ_g is denoted by the symbol $Norbit(E(\bar{x}^0), g)$.

Let $D(W)$ ($D(W_0)$) be the graph of transitions for Markov chain that is constructed on the states of the systems W (W_0).

Theorem A. For any marking R that satisfy conditions A1 - A2, i.e. for any potential system, the following conditions are true:

(1) for any $x \in W$ on the graph of transitions $D(W)$ there exists a directed sequence of finite length from x to W_0 ;

(2) W_0 is the only set of essential states, that is to say once the system enters this set it will never leave it.

Theorem B. 1. Suppose, the graph of relations Γ is not bipartite. Let i be an arbitrary fixed vertex. Then, for any marking R which satisfies the conditions A1 - A2 (for any potential system) and for any $z(x) \in M_0$ the following equalities hold: $F(z(x)) = z(b_i x), F^2(z(x)) = z(a_i x)$, where b_i is a solution of the characteristic equation

$$v^2 = a_i$$

2. If the graph of relations Γ is bipartite and i, j are arbitrary fixed vertices that belong to A_1 and A_2 then for any potential marking R that satisfies A1 and A2 and for any $z(x, y) \in M_0$, $F(z(x, y)) = z(b_{i,j} y, b_{j,i} x)$, where $b_{i,j}, b_{j,i}$ are solutions of the characteristic equations

$$v * w = a_i \quad w * v = a_j$$

($v = b_{i,j}, w = b_{j,i}$). Note, that $F^2(z(x, y)) = z(a_i x, a_j y)$

Example. Let the state of automaton E be a finite set and therefore a group of reactions be the group permutation of the set E . Every permutation g is a product of disjoint cycles and the number of the cycles in this decomposition is equal to $Norbit(E, g)$.

The dimension of the simplex of the stationary probability measures of the Markov chain minus one is equal to the number of the disjoint cycles in the decomposition of permutation v if Γ isn't bipartite or of permutation $\sigma : (x, y) \rightarrow (wy, vx)$ if Γ is bipartite.

According to **a principle of maximum nonergodicity** we have to find the solutions of the characteristic equations which maximize the number of the cycles in decomposition of itself or maximize the number $Norbit(E, v)$ (the maximum is taken over all solutions of the equation).

Conclusion. The most important case for is case when initial field of reactions R is product potential. In this case all characteristic reactions a_i equal e . If field R is product potential then field R^* (defined on the "two-steps" graph) is automatically product potential. From the principle of maximal nonergodicity best solution characteristic equation $v^2 = e$ ($v^*w = e$ and $w^*v = e$) is identical permutation e . In this case $F(z(x)) = z(x)$ ($F(z(x, y)) = z(x, y)$) and number of stationary measures equal number stable points of transformation F (see Model 1). It is mean that for product potentiality of initial field of reaction set W is set of stable points of function F . So we have deduced all property of Model 1 from Theorem B and "principle of maximal nonergodicity".

Next will find the general solution to the existence problem for heterogeneous potential systems.

Part 2. THE SOLUTION OF THE EXISTENCE PROBLEM FOR TWO DIMENSIONAL HETEROGENEOUS PRODUCT POTENTIAL SOCIAL SYSTEM

2.1. Set up of problem and definitions.

The social system or network is oriented graph (graph of relations), where any node represents the person and any edge represents relations. If any edge of social system marked by elements of algebraic group G (group of psychological reactions on the type of relations, where $g^2 = e$) and if product all elements along arbitrary closed path in the order in which the path goes equals unit element, then social system has called the social potential system or network.

Very attractive way to solve existing problem is just convert problem for discrete object (social network) to the similar problem on a rigid medium. Why? Because for the solid domain we can use calculus and mains concepts of theoretical physics: non-Abelian fields, infinitesimal equations for fields and so on.

The main problem that was solved in this article was problem of the existence of social potential marking (fields). For this purpose was created special method by using the smooth potential fields on a rigid medium. For smooth potential fields we wrote the system of infinitesimal equations that must hold for all potential fields. It is a system of partial differential equations that was transformed into the system of linear equations with one additional condition on the solution: the matrix-solution and field have to be an anti-commutative pair.

Then we found solutions of infinitesimal equation: the solution is any parameterizations of intersection of second degree surface (set of matrices that $A^2 = E$) and arbitrary plane.

The set of solution can be represented in the few canonical forms. Then the property of potentiality was checked by computer calculation for all types of potential fields.

Right now we define the product or path ordered integral for the solid domain.

A path ordered integral for non-Abelian fields (P-integral) can be defined as

$$P \left[\int A dx \right] = \lim_{n \rightarrow \infty} \prod_{i=1}^n (A(x_i) \delta x_i)$$

, where the product goes along the path in the order in which the path goes.

The properties of P-integral see [31].

The physicist use the Pexp path ordered integral for non-Abelian fields:

$$Pexp \int A dx = \lim_{n \rightarrow \infty} \prod_{i=1}^n (1 + iA(x_i) \delta x_i)$$

, where the product goes along the path in the order in which the path goes (see for instance [32]).

It is easy to see in our case that the definition of the integral depends from what the kind of integer n will be taken. If all number are even we get one result; for odd number we get completely different result: the determinant of A is -1 and the determinant of product the even (odd) number of matrices is $+1$ (-1). If we want to use P-integral as tool for theory of potential system, the number of step (n) must always be even integer number and we call it P_2 - integral. If number steps n is odd then we will receive P_1 - integral. We will use both.

2.2 The example of solution of the existence problem for discrete potential system.

We give the whole solution of this problem for group of reactions (transformations) $G = \{g, e\}$, where $g^2 = e$ and full graph of relations. The solution will be done by algorithm. This algorithm generate the family of G -potential fields (marks) or potential fields $\{g(j,k)$, where $j,k = 1, \dots, N\}$ on the full graph of relations with N notes, where N bigger than 3.

1st step. Let us put $g(1,2)$ equals g or e , $g(1,3)$ equals g or e and $g(2,3) = g(1,2)g(1,3)$. Let $g(2,1) = g(1,2)$, $g(3,1) = g(1,3)$, $g(3,2) = g(2,3)$.

2nd step. Let us put $g(1,4)$ equals g or e and $g(2,4) = g(1,2)g(1,4)$. $g(3,4) = g(1,3)g(1,4)$.

Step m . Let us put $g(1,m)$ equals g or e , $g(2,m) = g(1,2)g(1,m)$, $g(3,m) = g(2,3)g(2,m)$, \dots , $g(j,m) = g(j-2,j-1)g(j-2,m)$, \dots , $g(m-1,m) = g(m-2,m-1)g(m-2,m)$.

It is easy to prove that all this fields (marks) are potential and all potential fields (marks) can be generated by this algorithm.

Examples. The mark $g(1,k) = g(k,1) = g$, where $k = 2, \dots, N$ and $g(s,j) = g(j,s) = e$, where $s, j = 2, \dots, N$. More interesting example: $g(1,2)=g$, $g(1,3)=e$, $g(1,4)=g$, and so on. Let $g(2,3)=g$, $g(2,4)=e$, $g(2,5)=g$ and so on. In general case $g(k,k+1)=g$, $g(k,k+2)=e$, $g(k,k+3)=g$, $g(k,k+4)=e$, and so on.

2.3. The general description of the field (marking) $A(x)$ and properties of product N matrices.

For a heterogeneous potential system we have to find general description of the field (marking) $A(x)$, where $A(x)^2 = e$, $A(x)$ belongs to group of psychological reactions for all x , and x is vector of parameters and when our group is the two dimensional group of the matrices. It is easy to see that the arbitrary 2 by 2 matrix A satisfy this condition if and

only if $A = GZG^{-1}$, where G is arbitrary element of $GL(2)$ and

$$Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is also easily seen that $Z^2 = E$ and $A^2 = GZG^{-1}GZG^{-1} = GZZG^{-1} = GZG^{-1} = A$. For elements $a(i,j)$ of the matrices $A = GZG^{-1}$ the next property hold: $a(1,1) = -a(2,1)$ and $a(1,2)a(2,1) = 1 - a(1,1)^2$. We then to deduce by direct calculation: $a(1,1) = (bd - ac)/(ad - bc)$, $a(1,2) = (a^2 - b^2)/(ad - bc)$, $a(2,1) = (d^2 - c^2)/(ad - bc)$, $a(2,2) = (ac - bd)/(ad - bc)$, where

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\det G = ad - bc$ not equal zero.

This means that the set of 2-dimensional matrices that its square is unit matrix is a two dimensional manifold in three dimensional space $D = \{A(a,b,c) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \text{ where } bc = 1 - a^2\}$.

Suppose we have matrix $A(a,b,c)$, where $bc = 1 - a^2$. How do we find matrix G that $A(a,b,c) = G * Z * G^{-1}$? For solving this problem we have to find solution the linear system $A(a,b,c)G = GZ$.

Let $G = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then solution $G = \begin{pmatrix} x & ax + bz \\ z & cx - az \end{pmatrix}$ or

$$G = \begin{pmatrix} ay + bw & y \\ cy - aw & w \end{pmatrix}.$$

So we can rewrite $G = [x, Ax]$ or $[Ay, y]$, where vector x (y) represent first (second) column of matrix G .

The general solution is $G = [x, Ax]$ ($G = [Ay, y]$) for arbitrary vector x (y) and inequality $\det(G)$ not zero must hold. How easy to see $\det[x, Ax] = 0$ if and only if $x = 0$ or $Ax = cx$ for nonzero x , where c is constant. The eigenvalues of matrix A are $+1$ and -1 and eigenvectors are $e(1) = \begin{pmatrix} -b \\ a - 1 \end{pmatrix}$, $e(2) = \begin{pmatrix} -b \\ a + 1 \end{pmatrix}$.

So, the $\det(G)$ is nonzero if and only if vector x (y) does not equal 0 (zero vector) or not to be proportional to eigenvector $e(1)$ or eigenvector $e(2)$.

Then we have to find the commutator of matrix Z : $\text{Com}(Z) = \{S: SZ = ZS\}$.

In two dimensional case the general element of $\text{Com}(Z)$ can be represented in form $S = aE + bZ$, where a and b are arbitrary numbers and E is unit matrix. It means that GZG^{-1} and $(GS)Z(GS)^{-1}$ are same for all S from $\text{Com}(Z)$.

The product of two matrices $A(1)A(2)$ can not belong to set

$$D = \{A = GZG^{-1} \text{ for all } G \text{ from } GL(2)\}$$

because $\det(A) = -1$, but $\det(A(1)A(2)) = +1$.

Then $A(1)A(2)$ can be equal unit matrix E if only if $A(1) = A(2)$. So, the product only odd number of elements from D can belong to D .

2.4 The infinitesimal equations for social potential fields and solution of the existence problem for general two-dimensional social potential system on the solid set.

Let us define set of 2 by 2 matrices $A(a,b,c)$ that square equal E as $A(a, b, s) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $bc = 1 - a^2$.

Then take the square $[0,1] \times [0,1]$ on the plane xOy . Let $h=1/n$, where n is integer number, and divide the square on the n^2 small squares with length of side equal h . Then divide any small square on the two triangles by main diagonal. Suppose we three smooth functions $f_1(x,y), f_2(x,y)$, and $f_3(x,y)$ defined on the square $[0,1] \times [0,1]$, where $f_2(x,y)f_3(x,y) = 1 - f_1(x,y)^2$. In this case we automatically define set of matrices $A(f_1(x,y), f_2(x,y), f_3(x,y))$ on the square $[0,1] \times [0,1]$. Let us take the triangle grid with step h and define the network of mark of edges by matrices $A(f_1(x,y), f_2(x,y), f_3(x,y))$ taken in middle points of edge. We can start from interval $y=0$ and $0 \leq x \leq 1$. We will find condition when heterogeneous (non-homogeneous) distribution is potential. It means that product integrals along two curves started in same point and ended in same point are equal. It means that we have to find condition on the $f_1(x,y), f_2(x,y), f_3(x,y)$: $M(c_1(C,D), A(f_1(x,y), f_2(x,y), f_3(x,y))) = M(c_2(C,D), A(f_1(x,y), f_2(x,y), f_3(x,y)))$, where C and D are arbitrary points on the square $[0,1] \times [0,1]$ and c_1 and c_2 are curves connected C and D .

Lemma. The social fields with values in the set of matrices $A(f_1(x,y), f_2(x,y), f_3(x,y))$ is potential if and only if satisfy the system of differential equations):

$$A(d^2 A/dxdy) + dA/dy dA/dx = 0$$

, where $A^2 = A$ and $f_2(x,y)f_3(x,y) = 1 - f_1(x,y)^2$.

Proof. Let use next notation $dA/dx = A_x$, $dA/dy = A_y$, $d^2 A/dxdx = A_{xx}$, $d^2 A/dxdy = A_{xy}$, $d^2 A/dydy = A_{yy}$ and so on.

We have to calculate the production of four matrices $A(x, y+dx)A(x+dx, y+dy)A(x+dx, y)A(x, y)$ and equate the E. But $A(x, y+dx)A(x+dx, y+dy)A(x+dx, y)A(x, y) = E + (AA_x + A_x A)dx + (A A_y + A_y A)dy + ((1/2)A A_{xx} + A A_x A_x + ((1/2)A_{xx} A)d^2 x + (A A_{xy} + A A_y A_x + A_y A_x + A_y A A_x A)dxdy + ((1/2)A A_{yy} + A_y A_y + ((1/2)A_{yy} A)d^2 y + \dots = E + (AA_{xy} + A_y A_x)dxdy + \dots$. So for all dx and dy the equality $E + (AA_{xy} + A_y A_x)dxdy = E$ hold. But it is possible if and only if $AA_{xy} + A_y A_x = 0$. For calculation we have widely used next property: $AA = E$. It means that $AA_x = -A_x A$, $A_{xx} A + 2A_x A_x + AA_{xx} = 0$, $A_{xy} A + A_x A_y + A_y A_x + A A_{xy} = 0$ and so on.

The social potential marking (field) have to be solution to nonlinear partial differential equation $AA_{xy} + A_y A_x = 0$ or $A_{xy} A + A_y A_x = 0$. The equation is actually the system of linear ODE: $(AA_x)_y = A A_{xy} + A_y A_x$ and $(AA_y)_x = AA_{xy} + A_x A_y$. So our equations are

$$(A_x A)_y = 0 \text{ and } (A A_y)_x = 0.$$

The last equations mean that

$$A_x A = B(x) \text{ and } A A_y = D(y),$$

where $B(x)$ ($D(y)$) is matrix-function only from one variable x (y) and equality $AB = -BA$ ($AD = -DA$) hold.

We can get more information about the solid potential system if transform it by using exponential representation for A : $A = \exp(B)$. For this purpose we will use the spectral representation for A . We calculate $A = Z(1) - Z(2)$ and arbitrary function $f(x)$ $f(A) = f(1)Z(1) + f(2)Z(2)$, where $Z(1) = (1/2)(A + E)$, $Z(2) = (-0.5)(A - E)$, $Z(1) + Z(2) = E$, $Z(1)^2 = Z(1)$, $Z(2)^2 = Z(2)$, $Z(1)Z(2) = Z(2)Z(1) = 0$. If $A = \exp(B)$, where B is unknown matrix,

then $B = \ln(A) = \ln(1) Z(1) + \ln(-1) Z(2)$, where $\ln(1) = 0$, $\ln(-1) = \pi + 2\pi ki$, $i^2 = -1$ and k is arbitrary integer. It means that

$$B = \pi Z(2).$$

We now easily to calculate $\exp(B) = E - Z(2)\cos(p) Z(2) + \sin(p) Z(2)i = Z(1) - Z(2) = A$.

The matrices $A(a,b,c)$ can be rewrite in more familiar form. Let put $b = x + iy = z, c = x - iy = \bar{z}$. In this case we get $A(a,z) = A(a,x,y) = \begin{pmatrix} a & z \\ \bar{z} & -a \end{pmatrix}$,

where $a^2 + x^2 + y^2 = 1$ ($x = (b + c)/2$, $y = (b - c)/i2$ and $\det A = 1$).

We can represent the field as matrix dependent from one complex potential function:

$$A(z) = \begin{pmatrix} \sqrt{1 - z\bar{z}} & z \\ \bar{z} & -\sqrt{1 - z\bar{z}} \end{pmatrix}$$

, where $z = z(u,v) = x(u,v) + iy(u,v)$ is an arbitrary potential function on the square $[0,1] \times [0,1]$. So we have the number potential system equal the number of potential function on the square.

2.5 General and particular solutions of 2-dimensional infinitesimal equation.

The system of differential equations (infinitesimal condition for potentiality) $AA_{xy} + A_yA_x = (A_xA)_y = 0$ and $(AA_y)_x = 0$ are really is first order system of ODE.

What follow is the general description of all non-constant solution. We will start from $AA_y = C(y)$, where $C(y)$ is matrix-function only from one variable y and equality $AC = -CA$ hold. So first of all we have to describe all matrices C that $AC = -CA$. Really we will describe a set of $A(a,b,c)$ where $AC = -C$: the (a, b, c) must satisfy equation $2aC_1 + cC_2 + bC_3 = 0$ for any matrix

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_1 \end{pmatrix}.$$

It means that all triplet number C_1, C_2 , and C_3 we can find solution by solve system of algebraic equations: $a^2 + bc = 1$ and $2aC_1 + cC_2 + bC_3 = 0$.

It means that (a, b, c) have to satisfy quadric equation: $c^2(C_2^2 + 2bc(C_2C_3 + 2C_1^2)) + b^2C_3^2 = 4C_1^2$.

We can then prove by direct calculation the next theorem.

Theorem. For any triplets $(C_1(y), C_2(y), C_3(y))$ there are solutions of equation $AA_y = C(y)$ that can be found as parameterization of the intersection of hyperbolic $a^2 + bc - 1 = 0$ and plane $2aC_1 + cC_2 + bC_3 = 0$. Similarly we can describe the set of solutions of equation $A_xA = B(x)$.

Surprisingly, that to get the normal two variable fields at is easier use 3 by 3 matrices.

Examples of solutions. For instance, if C_1 equal zero ($a^2 + bc = 1$, $cC_2 + bC_3 = 0$) and $C_2C_3 > 0$ then our equation represent the hyperbola $a^2 - b^2C_3/C_2 = 1$. Let us put $a(t) = \cosh(t)$, $b(t) = \sqrt{C_2/C_3} \sinh(t)$.

In this case

$$A = \begin{pmatrix} \cosh(t) & \sqrt{C_2/C_3} \sinh(t) \\ -\sqrt{C_2/C_3} \sinh(t) & -\cosh(t) \end{pmatrix}$$

, where $t = t(x, y)$. $A_t = \begin{pmatrix} \sinh(t) & \sqrt{C_2/C_3} \cosh(t) \\ -\sqrt{C_2/C_3} \cosh(t) & -\sinh(t) \end{pmatrix}$ and $A_y = A_t t_y$. Therefore

$$AA_y = t_y \begin{pmatrix} 0 & \sqrt{C_2/C_3} \\ \sqrt{C_3/C_2} & 0 \end{pmatrix}.$$

So our equation $AA_y = C$, where $C = \begin{pmatrix} 0 & C_2 \\ C_3 & 0 \end{pmatrix}$ can be transformed into system $t_y \sqrt{C_2/C_3} = C_2$, $t_y \sqrt{C_3/C_2} = C_3$ or it means that really we have one equation $t_y = \sqrt{C_3 C_2}$.

And after integration $t = \int \sqrt{C_3 C_2} dy + R(x)$, where $R(x)$ is an arbitrary function from x or constant ($C_2 = C_2(y)$, $C_3 = C_3(y)$ are function of y or constants). In case when C_1 equal zero and $C_2 C_3 < 0$ we have an ellipse and solution

$$A = \begin{pmatrix} \cos(t) & \sqrt{-C_2/C_3} \sin(t) \\ -\sqrt{-C_2/C_3} \sin(t) & -\cos(t) \end{pmatrix}$$

, where $t = \int \sqrt{C_3 C_2} dy + R(x)$.

Suppose C_1 is nonzero function or constant. In this case we have

$$\begin{pmatrix} -\cosh(t) + \sinh(t) & L^{-1}(-\sinh(t) + \cosh(t)) \\ 2L \sinh(t) & \cosh(t) - \sinh(t) \end{pmatrix}$$

or

$$A = \begin{pmatrix} \cosh(t) & L^{-1} \sinh(t) \\ -L \sinh(t) & -\cosh(t) \end{pmatrix}.$$

Numerical calculation. We can use numerical calculation for checking of the potentiality of fields (see previous theorem and calculations). Suppose

$$A(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix},$$

where $0 \leq t \leq 1$ and $A(t)$ is the solution of an infinitesimal equation. The parameterization of interval $[0, 1]$ can be define as function $t = f(x)$, where x belong some interval $[a, b]$. The P-integral from potential field must be independent from parameterization: $P \left[\int A dx \right] = P \left[\int A dy \right]$ for two arbitrary parameterizations $t = f(x)$, $a \leq x \leq b$, $f(a) = 0$, $f(b) = 1$, $t = g(y)$, $c \leq y \leq d$, $g(c) = 0$, $g(d) = 1$ or close P-integral must be equal E (x goes from a to b and then y goes from d to c). We found by numerical calculation this property for different parameterizations: $f(x) = \sin(x)$, $0 \leq x \leq \pi/2$, $g(y) = y^m$, $0 \leq y \leq 1$, $m = 2, 3$ and so on.

2.6 Final step: transformation continuous case into discrete.

We can use D- fields and (E, D) - fields defined in solid space. (E,D) - fields can be generated from D-fields by chose finite number points and then "over blowing" points in domains with values on the bounds of domain equal to value in given points. Then we must just define field equal to the E for all internal points of domains.

When product potential D-fields will be chosen we have to immerse graph into solid space and, if we wont get (E, D) - field, "over blowing" same nodes of immersed graph. Then for getting discrete values of field on the edges we have to take P1 (limit for sets of partitions of odd number points on edge) or P2 (limit for sets of partitions of even number points on edge) product integral along all edges. But for every close path on the graph the number edges where we were taken P1 integral must be even! Because of E - fields are product potential for P2- product integral. We can you different combination of P1 and P2 integral

along different edges. Some times this procedure gives us many combinations of product potential system of marking of edges.

Example.

For a triangle we can get only two combinations: for all three edges were take P_2 - integral and for two edges were take P_1 - integral and for one edge was take P_2 - integral.

We define the function of relation $f(k, j) = \det G(k, j)$, where $G(k, j)$ equal P_1 or P_2 integral for D- or (E,D)- field along edge (k, j) . It is clear that to us that number negative values of relation equal the number edges with negative determinant of reactions, but this number must be positive.

So, all product potential systems automatically are balanced systems.

And for product potential system structural theorem for full graph of relation is true: for product potential system exist a maximum two antagonistic groups.

How arbitrary D field can be transformed into product potential field?

Mechanism look very easy for D - field defined on the solid domain. What does it mean to product potential (we use P_2 product integral)? It means that all values must belong to plane $Aa + Bb + Cc = 0$ (where $A=2C_1$, $B=C_3$, $C=C_2$) in given domain. So we have systems $a^2 + bc = 1$ and $Aa + Bb + Cc = 0$, where (a, b, c) belongs to small domain, that really is ellipse or hyperbola. Then we make gradient system on the surface $a^2 + bc = 1$ that has direction to curve - ellipse or hyperbola ($a^2 + bc = 1$ and $Aa + Bb + Cc = 0$).

The our program has been realized. We can submerge the graph without intersection into N- dimensional space and then we can find the non-constant social potential fields defined in the same N-dimensional Euclidean area.

Part 3. THE DYNAMIC OF PRODUCT POTENTIAL SOCIAL SYSTEM.

In part three we will define a set of special "control matrices" M and a generating set of finite words M^* in the alphabet M . Simultaneously, we will define a set of stochastic matrices M_s and a set of finite words M_s^* . Then we will define an product process on the set of stochastic and control matrices. Our stochastic process will be a left-side multiplier acting on a randomly chosen control matrix. Then we will have to make the final step for the description of the process: every finite product of matrices (words in alphabet M) has to be multiplied by a special matrix R_g in which all elements are psychological reactions. We will use a matrix representation for psychological reactions and R_g will be a matrix with matrix elements. Multiplication on the R_g will be defined in a special way too: we will use symbol $*$ multiplication.

The set of words in an alphabet M contains some special element: the so-called "left ideal" matrices. Products of control matrices M generate all left ideal matrices. The ideal matrices have a very important property: when an arbitrary stochastic/control matrix is multiplied from the left by an ideal matrix one obtains a "left ideal" matrix. So the set of "left ideal" matrices is the termination set for our stochastic product process. It means that once the system reaches the termination set the process can never leave the termination set. Thus the "left ideal" matrices play a crucial role in the description of our process.

The knowledge of the structure of M^* allows us to prove the main theorem for a product potential system: there is a one to one connection between distinct ideals and invariant measures.

3.1. The algebraic description of the dynamics of the states for potential systems.

The dynamics of the states of a net of automata was determined previously [33-36] by a family of conditional probabilities. But in the case when group of relations is a connected graph we will find the algebraic description of the dynamics of the states. In the models, which will be presented in this paper, the dynamics of states are controlled by a subset of a stochastic matrices so call set of "control matrices". The set of stochastic matrices M_s are matrices with the integer entries equal to 0 or 1 and with sum of all rows equal one.

Let $M(B)$ is set of control matrices, where control matrices are stochastic matrices with zero elements on the main diagonal ($\text{Tr}(A)=0$) and all elements less or equal to elements of adjacency matrix B of original connected graph. Elements of $M(B)$ we will call *the control matrices*.

Let $M(B)^*$ is set of finite length words in alphabet $M(B)$. So $M(B)^*$ is set of words $A_m A_{(m-1)} \dots A_1$, where A_1, A_2, \dots, A_m is matrices from $M(B)$. So "word" is productions of m matrices for any not negative m . The sets of words $M(B)^*$ is algebraic semigroups.

Matrix of reaction R_g presents the product potential system reactions on the complete graph: $g(k,l)$ is reactions on the edge (k,l) and $g(s,s)=e$, where e is identical transformation ($ex=x$ for all x). Potentiality on the complete graph means that $g(k,l)g(l,s)g(k,s) = e$, $g(k,l)g(l,s)=e$ for any three nodes k, l, s . Therefore, $g(k,l)g(l,s)g(k,s) = e$ because of property of potentiality and $gg=e$.

We will show that local interacting process for product potential system of reactions can be defined as algebraic presentation set of control matrices into semigroup of transformations.

The representation is map F of algebraic semigroup H into the group of liner matrices GL , where for any two arbitrary elements from H property $F(h_1 h_2) = F(h_1)F(h_2)$ hold ($F(h_1)$ and $F(h_2)$ are linear matrices).

Right now we show connection between old description of the dynamic and new one on the particular example.

Suppose all edges of graph of relations are marked by the elements belonging to the group of reactions G in accordance with the given system of reactions and the choice functions. For the arbitrary finite group of relations the system of elements marking a graph of relations can be written in the form of a square matrix. The dynamics of the system after n steps is determined by a product of n matrices applied to the initial state of the net, where every matrix is a $*$ - product of the control matrix and the square matrix of the transformations (every entry in this matrix is a transformation of the state space). If we are given a system of the conditional probabilities on the $*$ - product matrix, then the dynamics of the network's state will be defined.

Let $A = (a_{i,j})$, where $1 \leq i, j \leq n$ and $B = (b_{i,j})$, where $1 \leq i, j \leq n$. The matrix $A*B = (a_{i,j}b_{i,j})$, where $1 \leq i, j \leq n$ is called a $*$ -product of the matrices A and B .

Example . Suppose, the graph of relations Γ_n is complete and $n = 3$. Let $\bar{x}^0 = (x_1, x_2, x_3)$ is the initial state of the net and a marking (potential field) $R = \{g_{i,j}\}$, where $(i,j) \in B$ are given.

The few steps of the dynamics of the states of the system will be described below.

Step 1. We have $\bar{x}^0 \rightarrow F(\bar{x}^0) = \{g_{1,2}x_2, g_{1,3}x_3\} \times \{g_{2,1}x_1, g_{2,3}x_3\} \times \{g_{3,1}x_1, g_{3,2}x_2\}$. Suppose that the first automaton chose the state of the automaton number 2, second -

automaton number 3, third - automaton number 1 (the options are underlined) . It means that $\bar{x}^0 \rightarrow \bar{x}^1 = (g_{1,2}x_2, g_{2,3}x_3, g_{3,1}x_1)$.

Let matrix

$$Rg = \begin{pmatrix} e & g_{1,2} & g_{1,3} \\ g_{2,1} & e & g_{2,3} \\ g_{3,1} & g_{3,2} & e \end{pmatrix}$$

be the matrix of transformations and matrix

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

be the control matrix which reflects the options (the first automaton chose the state of automaton number 2 , second - automaton number 3 , third - automaton number 1).

We easily see that

$$C_1 * Rg = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} * \begin{pmatrix} e & g_{1,2} & g_{1,3} \\ g_{2,1} & e & g_{2,3} \\ g_{3,1} & g_{3,2} & e \end{pmatrix} = \begin{pmatrix} 0 & g_{1,2} & 0 \\ 0 & 0 & g_{2,3} \\ g_{3,1} & 0 & 0 \end{pmatrix},$$

and $C_1 * Rg\bar{x}^{0t} = \bar{x}^{1t}$, where \bar{x}^t denote the matrix transpose to the matrix \bar{x} . We can check it directly

$$C_1 * Rg\bar{x}^{0t} = \begin{pmatrix} 0 & g_{1,2} & 0 \\ 0 & 0 & g_{2,3} \\ g_{3,1} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} g_{1,2}x_2 \\ g_{2,3}x_3 \\ g_{3,1}x_1 \end{pmatrix} = \bar{x}^{1t}$$

We see that both the methods give us the same result.

Step 2. Second step is $\bar{x}^1 \rightarrow F(\bar{x}^1) = \{g_{1,2}x_2^1, g_{1,3}x_3^1\} \times \{g_{2,1}x_1^1, g_{2,3}x_3^1\} \times \{g_{3,1}x_1^1, g_{3,2}x_2^1\}$ and suppose the first automaton chose the state of automaton number 3 , second - automaton number 1 , third - automaton number 2 , that will be represented by the control matrix

$$C_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(Note that $x_1^1 = g_{1,2}x_2$, $x_2^1 = g_{2,3}x_3$, $x_3^1 = g_{3,1}x_1$).

It means that on the second step the state of the system \bar{x}^1 was transformed into the new state of the system $\bar{x}^2 = (x_1^2, x_2^2, x_3^2)$, where $x_1^2 = g_{1,3}x_3^1$, $x_2^2 = g_{2,1}x_1^1$, $x_3^2 = g_{3,2}x_2^1$.

Similarly to what we done in the step 1 we can find that

$$C_2 * Rg\bar{x}^{1t} = \begin{pmatrix} 0 & 0 & g_{1,3} \\ g_{2,1} & 0 & 0 \\ 0 & g_{3,2} & 0 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{pmatrix} = \bar{x}^{2t}$$

and $\bar{x}^{2t} = (C_2 * Rg)\bar{x}^{1t} = (C_2 * Rg)(C_1 * Rg)\bar{x}^{0t}$ and so on .

So we found that product potential system satisfy next very important equality

$$(C_2C_1) * Rg = (C_2 * Rg)(C_1 * Rg)$$

After n steps we will obtain the state of system

$$\bar{x}^{nt} = (C_n * Rg)(C_{n-1} * Rg) \dots (C_2 * Rg)(C_1 * Rg)\bar{x}^{0t}$$

, where $C_1, C_2, \dots, C_{n-1}, C_n$ are randomly chosen control matrices.

We call semigroups $A(M^*) = (M^*)^* Rg = \{ w^* Rg \text{ for all finite words from } M^* \}$ and $A(M_s^*) = (M_s^*)^* Rg = \{ w^* Rg \text{ for all finite words from } M_s^* \}$ semigroups of transformations (operators). The elements of matrices of transformations are reactions. For instances it will be 2x2 matrices (see Part 2).

3.2. The representation of the subgroup and description of the states dynamics.

The homomorphism $\rho : M^* \rightarrow (M^*)^* Rg$ is a representation of the semigroup M^* into the semigroup of operators $A(M^*)$, where $\rho(w) = w^* Rg$, for any word w from M^* .

Example. Let $C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $C_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $C_2 C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\text{Therefore } (C_2^* Rg)(C_1^* Rg) = \begin{pmatrix} 0 & g_{1,2} & 0 \\ 0 & 0 & g_{2,3} \\ 0 & g_{3,2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & g_{1,3} \\ g_{2,1} & 0 & 0 \\ 0 & g_{3,2} & 0 \end{pmatrix} = \begin{pmatrix} g_{1,2}g_{2,1} & 0 & 0 \\ 0 & g_{2,3}g_{3,2} & 0 \\ g_{3,2}g_{2,1} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ g_{3,1} & 0 & 0 \end{pmatrix}.$$

We use potentiality of field $g_{i,j}$: $g_{1,2}g_{2,1} = e$, $g_{2,3}g_{3,2} = e$, and $g_{3,2}g_{2,1} = g_{3,1}$.

$$\text{Contrariwise } (C_2 C_1)^* Rg = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} e & g_{1,2} & g_{1,3} \\ g_{2,1} & e & g_{2,3} \\ g_{3,1} & g_{3,2} & e \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ g_{3,1} & 0 & 0 \end{pmatrix}.$$

So $\rho(C_2 C_1) = (C_2 C_1)^* Rg = (C_2^* Rg)(C_1^* Rg) = \rho(C_2) \rho(C_1)$.

In general case very easy to prove that homomorphism $\rho : M^* \rightarrow A(M^*)$ ($\rho : M_s^* \rightarrow A(M_s^*)$) is a representation of the ring M^* (M_s^*) into the ring of operators ($A(M^*)$ ($A(M_s^*)$)).

We call set I **left ideal** if for any word w wI belongs to I ($wI \subset I$).

The next theorem will be base for main Theorem 2.

Note. All previous examples graph of relations are complete. But we can use this the algebraic method (the method of algebraic description of dynamic of the states for potential system) for arbitrary finite connected graph of relation marked by product potentials system of reactions. For this purpose we have to make our graph complete by adding new edges. Then we will mark new edges by reactions equal product of reactions along any path on the original graph started in source vertex of edges and ending in terminating vertex of edges. The set of control matrices contains only control matrices that less or equal to adjacency matrix of original graph. Similarly the entries of the reaction matrices Rg are reactions on original graph and new ones (built on the added edges).

Example of extended matrix of reactions Rg for connected, but not complete graph of reactions.

Suppose we have graph of relation (A, V) , where $A = \{ 1, 2, 3, 4 \}$ and set of edges $V = \{ (1,2), (2,1), (2,3), (3,2), (3,4), (4,3), (1,4), (4,1) \}$. Suppose that field of reactions is potential. It is mean that $g_{i,j}g_{j,i} = e$ for all edges from V and $g_{1,4}g_{4,3}g_{3,2}g_{2,1} = g_{1,2}g_{2,3}g_{3,4}g_{4,1} = e$. We add two pairs edges $\{ (4,2), (2,4), (1,3), (3,1) \}$ and expand initial fields on the new edges keeping new field potential. It is mean that $g_{2,4} = g_{2,1}g_{1,4}$, $g_{4,2} = g_{4,1}g_{1,2}$, $g_{1,3} = g_{1,4}g_{4,3}$, $g_{3,1} = g_{3,4}g_{4,1}$.

The matrix of reaction is

$$Rg = \begin{pmatrix} e & g_{1,2} & g_{1,4}g_{4,3} & g_{1,4} \\ g_{2,1} & e & g_{2,3} & g_{2,1}g_{1,4} \\ g_{3,4}g_{4,1} & g_{3,2} & e & g_{3,4} \\ g_{4,1} & g_{2,1}g_{1,4} & g_{4,3} & e \end{pmatrix}$$

The number of invariant measures will be equal to number of ideals as well. So in general case we have to study the semigroup of words $M^*(B)$ generated by set of control matrices $M(B)$, where all elements less or equal to elements of adjacency matrix B of original graph.

Theorem 1 1. Suppose that the graph of relations is not bipartite. The number of ideals of the semigroup $M^*(B)$ equals the number of the vertices and every ideal is generated by one control matrix $I_k = \{t_{k,j} = 1 \forall j \in \{1, 2, \dots, n\} \mid t_{i,s} = 0 \forall i \neq k \text{ and } \forall s\} \mid k \in \{1, 2, \dots, n\}$.

2. Suppose that the graph of relations is bipartite and $\Gamma = (A, V)$, where $A = A_1 \cup A_2, A_1 \cap A_2 = \emptyset$. The number of ideals of the semigroup $M^*(B)$ equals $|A_1| |A_2|$ and every ideal is generated by two elements $S(i, j)$ and $S^-(i, j)$, where $i \in A_1, j \in A_2$ and $S(i, j) = (s_{k,l}, \text{ where } s_{i,l} = 1, i \in A_1, \text{ for all } l \in A_1; s_{m,j} = 1, j \in A_2 \forall m \in A_2 \text{ and; all other elements are equal } 0)$, $S^-(i, j) = (s_{i,l} = 1, i \in A_1, \text{ for all } l \in A_2; s_{m,j} = 1, \text{ where } j \in A_2, \text{ for all } m \in A_1 \text{ and all other elements are } 0)$.

The sketch of proof of the theorem 1.

Suppose that the graph of relations $\Gamma = (A, V)$ is not bipartite, where $A = \{1, 2, \dots, n\}$. For arbitrary vertex i we will find set of vertices $A_1 = A(i)$ that connected with vertex i (for every j from A_2 exist edge (j, i) from V). Then we find second set $A_2 = \{j \mid \text{exist edge } (j, k), \text{ where } k \in A_1\}$ and so on. Throughout finite number steps we will reach the set A . So we have chain $\{i\} \leftarrow A_1 \leftarrow A_2 \dots \leftarrow A_{m-1} \leftarrow A_m = \{1, 2, \dots, n\}$. Every control matrix maps $A = \{1, 2, \dots, n\}$ into subset of A . Take matrix C_1 that transform $A_m = A$ into A_{m-1} , then take control matrix that transfer A_{m-1} into A_{m-2} and so on and on the last step take control matrix C_m that map A_1 into $\{i\}$. The product $C_1 C_2 \dots C_{m-1} C_m$ will be equal I_i .

Similar procedure can be used for getting ideals $\{S(i, j), S^-(i, j)\}$ when graph of relations is bipartite ($A = A_1 \cup A_2$). Only one difference: we will start from arbitrary pair (i, j) , where i belongs to A_1 and j belongs to A_2 .

Representation of the semigroup and dynamics of a state of a potential system.
Graph of transitions for the Markov chain.

Now we are ready to describe of the dynamics of a state of a potential system.

For describing the dynamics of a state of potential system the rings M^* and $A(M^*)$ and representations $\rho: M^* \rightarrow A(M^*)$ will be used. The dynamics of the states can be realized as representation of a random product process on the semigroup M^* into the semigroup $A(M^*)$. It means that first of all we get randomly chosen initial word (matrix) C_0 from M^* . Then we multiply initial word (matrix) on the randomly chosen matrix from M and so on. All steps are independent. Throughout n steps we get random trajectory $C_0, C_1 C_0, C_2 C_1 C_0, \dots, C_n C_{n-1} \dots C_1 C_0$ as a result of the random product process on the semigroup M^* . Then we map our trajectory on the semigroup M^* into the ring $A(M^*)$ and get real trajectory

$$\rho(C_0), \rho(C_1 C_0), \dots, \rho(C_n C_{n-1} \dots C_1 C_0)$$

, where $\rho(C_n C_{n-1} \dots C_1 C_0) = (C_n * Rg)(C_{n-1} * Rg) \dots (C_0 * Rg) = \rho(C_n) \rho(C_{n-1}) \dots \rho(C_0)$.

For an arbitrary initial word (matrix) with probability one we have to reach one of the ideals of the ring M^* . Therefore, according to theorem 1 we must examine the behavior of our system in the ideals. It hardly simplifies our problem.

Theorem 2. The number of ideals of the semigroup $A(M^*)$ equals to the number of ideals of the semigroup M^* .

Proof of the theorem 2. The theorem 1 means that stochastic product process on M^* with probability one converges to one of n ideals for non bipartite graph or $|A_1||A_2|$ ideals for bipartite graph.

Suppose we have non bipartite graph of relation. We map our trajectory that converge to ideal I_k into ring $A(M^*)$. The map ρ is representation and image of ideals I_1, \dots, I_n are ideals in $\rho(A(M^*))$: $\rho(\mathbf{w})\rho(I(k)) = \rho(\mathbf{w}I(k)) = \rho(I(k))$. It means that $\rho(I(1)), \dots, \rho(I(N))$ are left ideals of $\rho(A(M^*))$. The ideal $\rho(I(k))$ can be reached for finite number of steps. The ideals $\rho(I(k))$ represent final classes and system has to reach one of N final classes and never leave them.

The prove of theorem 2 for bipartite graph absolutely similar to previous one.

Connection between ideals of semigroup $A(M^*)$ and number of stationary measure of Markovs chain defined on the system's states.

Suppose we have product potential system and let denote by symbol Z a set of system's states. It is to easy to see that the set of final states W of Markov chain defined on the Z is union of set $\mathbf{I}Z$ for all ideals of semigroup $A(M^*)$ ($W = \bigcup \mathbf{I}Z$ for all ideals $A(M^*)$).

The number of stationary measures equal the number of elements of final state W (see Part 1).

Example (See Part1, Model 1, example 2) Suppose we have product potential field $f = \{g_{1,2} = g_{1,3} = g, g_{2,3} = e \text{ and } g_{i,j} = g_{i,j}\}$ defined on the complete graph of relation Γ_3 . Suppose that $X = \{-1, +1\}$ and $e, g: X \rightarrow X$, where $gx = -x$ and $ex = x$ for all x .

In this case semigroup $A(M^*)$ has three ideals $I_1^*Rg = \begin{pmatrix} e & 0 & 0 \\ g & 0 & 0 \\ g & 0 & 0 \end{pmatrix}$, $I_2^*Rg = \begin{pmatrix} 0 & g & 0 \\ 0 & e & 0 \\ 0 & e & 0 \end{pmatrix}$, $I_3^*Rg = \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & e \\ 0 & 0 & e \end{pmatrix}$, where $I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $I_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ are

ideals of M^* , and $Rg = \begin{pmatrix} e & g & g \\ g & e & e \\ g & e & e \end{pmatrix}$ is transformation matrix. The set of system's states

Z is 8 elements set $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ for all } x_1, x_2, x_3 \in X = \{-1, +1\} \right\}$.

All three set $(I_1^*Rg)Z$, $(I_2^*Rg)Z$, $(I_3^*Rg)Z$ are identical sets $((I_1^*Rg)Z \equiv (I_2^*Rg)Z \equiv (I_3^*Rg)Z)$. For instance $(I_1^*Rg)Z = \left\{ \begin{pmatrix} e & 0 & 0 \\ g & 0 & 0 \\ g & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} \equiv \left\{ \begin{pmatrix} x_1 \\ -x_1 \\ -x_1 \end{pmatrix} \right\}$
 $= x_1 \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix} \text{ for arbitrary } x_1 \text{ from } X \equiv \left\{ \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ +1 \\ +1 \end{pmatrix} \right\}$ is two element set and $W = (I_1^*Rg)Z \cup (I_2^*Rg)Z \cup (I_3^*Rg)Z \equiv (I_1^*Rg)Z$ contains exactly two elements.

So Markov chain defined on the system's states has exactly two stationary measures (or two thermodynamic states).

Conclusion.

A very natural assumption of social systems being locally interacting systems with relations and psychological reactions together with the principle of maximal nonergodicity enable us to show that a product potential system is a balanced (stable) system in the social sense.

This principle of maximal nonergodicity selects exactly product potential systems out of all locally interacting systems of automata with relations and reactions. Maximal non-ergodicity means that only systems (networks of automata) with this property have a maximum possible number of states (the state in thermodynamics is defined by the measure) for all possible relations. So the principle of maximal nonergodicity can be called the principle of maximal freedom for a group.

The system are balanced in social sense if the set of psychological reactions on the graph of relation satisfy the principle of maximum non-ergodicity and this system of reactions are product potential on the so call "two-steps" graph of relations. In real life survive only relatively stable groups and we can observe only stable groups that in social science call balanced. So reason why social system (group) is stable based on hidden potentiality of reactions: only social systems with potential system of reactions (potential fields) are stable (balanced in social sense). This conclusion is not a big surprise for natural (physical) systems. For instance the system consisting of a star and a single planet is stable because the gravitational interaction is potential (friction is absent). The potentiality of gravitational interaction means that work done along any closed path is zero. But for social science and, particularly for human groups, a similar property comes as a surprise.

Therefore, the reason why balanced groups (systems) can exist forever is a hidden potentiality of human reactions inside balanced groups. This is the main reason this system was studied in this article.

The main difficulty was the problem of existence for heterogeneous product potential systems. This problem has been solved by constructing infinitesimal differential equations. The problem of existence was completely solved for a smooth potential by solving infinitesimal differential equations. All product potential fields on the solid domain were found as solutions of infinitesimal differential equations for two-dimensional matrices.

Then a graph of relations was embedded in the domain. We proceeded by describing all product potential marks on the graph of relations by integrating continuous product potential fields along the graph's edges. This procedure used two different types of product integrals: P_1 and P_2 .

We proved that any product potential system on the graph is a balanced system in the social sense. Therefore, we can say that in our model being balanced is identical to it being a product potential system. So the structural theorem for a balanced group (stating that a balanced group has at most two antagonistic subgroups) is true for all product potential systems containing full graphs of relations.

We have proved that a locally interacting process for a product potential system of relations can be represented as an process of multiplication on a randomly chosen control matrix with a transformation into the original process in the end. The main property of a product potential system concerning the number of stable measures (there is a one to one connection between ideals and invariant measures) was derived.

This article proposes a bridge between certain concepts of natural sciences and sociology.

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